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MATB44 Week 7 Notes

1. Variation of Parameters:
- Another way to solve n-h eqns.
- Unlike methods of undetermined coefficients, this works for any RHS.

E.g. 1 Solve $y'' + y = \frac{1}{\sin t}$

Soln:

1. First, solve the homogeneous eqn.

$y'' + y = 0$
$r^2 + 1 = 0$
$r^2 = -1$
$r = \pm\sqrt{-1}$
$= \pm i$

$\lambda = 0,\ u = 1$

$y_1 = e^{\lambda t}\cos(ut)$ $\quad$ $y_2 = e^{\lambda t}\sin(ut)$
$= \cos(t)$ $\quad$ $= \sin(t)$

2. Let $y = u_1(t)y_1 + u_2(t)y_2$

$y' = (u_1y_1 + u_2y_2)'$
$= u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2'$

Note: To find u_1 and u_2, we need 2 eqns. The first is orthogonality.

(1) $u_1' y_1 + u_2' y_2 = 0$

Now, we have: $y' = u_1 y_1' + u_2 y_2'$

$y'' = (u_1 y_1' + u_2 y_2')'$

$= u_1' y_1' + u_1 y_1'' + u_2' y_2' + u_2 y_2''$

Going back to the original eqn and substituting the values for y'' and y, we get

$$\underbrace{u_1' y_1' + u_1 y_1'' + u_2' y_2' + u_2 y_2''}_{y''} + \underbrace{u_1 y_1 + u_2 y_2}_{y} = \frac{1}{\sin t}$$

Collect all the terms with u_1 and u_2.

$u_1 (y_1'' + y_1) = 0$

$u_2 (y_2'' + y_2) = 0$

We now have

$u_1' y_1' + u_2' y_2' = \frac{1}{\sin t}$ (2)

We now have our 2 eqns to find u_1 and u_2.

(1) $u_1' y_1 + u_2' y_2 = 0$

(2) $u_1' y_1' + u_2' y_2' = \frac{1}{\sin t}$

Recall that $y_1 = \cos t$ and $y_2 = \sin t$

$u_1' \cos t + u_2' \sin t = 0$
$-u_1' \sin t + u_2' \cos t = \frac{1}{\sin t}$

From $u_1' \cos t + u_2' \sin t = 0$
$\rightarrow u_1' \cos t = -u_2' \sin t$
$\rightarrow \frac{-u_1' \cos t}{\sin t} = u_2'$

Take this value of u_2' and plug it into the second eqn.

$$-u_1' \sin t - \frac{u_1' \cos^2 t}{\sin t} = \frac{1}{\sin t}$$

$-u_1' (\sin^2 t + \cos^2 t) = 1$
$-u_1' = 1$
$u_1' = -1$
$\int u_1' \, dt = \int -1 \, dt$
$u_1 + C_1 = -t + C_2$
$u_1 = -t + C_1'$

$$u_2' = \frac{-u_1' \cos t}{\sin t}$$
$$= \frac{\cos t}{\sin t}$$

$$\int u_2' \, dt = \int \frac{\cos t}{\sin t} \, dt$$

$u_2 = \log|\sin t| + c_2'$

$y = u_1 y_1 + u_2 y_2$
$= (-t + c_1')\cos t + (\log|\sin t| + c_2')\sin t$

↑ General Soln

Eg. 2 Solve $x^2 y'' + xy' + (x^2 - 0.25)y = 3x^{3/2}\sin x$
with $y_1 = x^{-1/2}\sin x$ and $y_2 = x^{-1/2}\cos x$

Soln:
1. If the coefficient of y'' is not 1, divide both sides of the eqn by it.

$$y'' + \frac{y'}{x} + \left(1 - \frac{0.25}{x^2}\right)y = 3x^{-1/2}\sin x$$

Let $y = u_1 y_1 + u_2 y_2$

(1) $u_1' y_1 + u_2' y_2 = 0$
(2) $u_1' y_1' + u_2' y_2' = 3x^{-1/2}\sin x$

From (1)
$$u_2' = -\frac{u_1' y_1}{y_2} = \frac{-u_1' x^{-1/2}\sin x}{x^{-1/2}\cos x} = -u_1' \tan x$$

Plug this into (2).

$u_1' y_1' + (-u_1' \tan x) y_2' = 3x^{-1/2} \sin x$

$u_1' (-\frac{1}{2} x^{-3/2} \sin x + x^{-1/2} \cos x) -$

$u_1' (-\frac{1}{2} x^{-3/2} \cos x - x^{-1/2} \sin x)$

$(\tan x) = 3x^{-1/2} \sin x$

$u_1' (-\frac{1}{2} x^{-3/2} \sin x + x^{-1/2} \cos x) -$

$u_1' (-\frac{1}{2} x^{-3/2} \sin x - x^{-1/2} \frac{\sin^2 x}{\cos x}) = 3x^{-1/2} \sin x$

$$u_1' \left(\cancel{-\frac{1}{2} x^{-3/2}} \sin x + x^{-1/2} \cos x + \cancel{\frac{1}{2} x^{-3/2}} \sin x + x^{-1/2} \frac{\sin^2 x}{\cos x} \right)$$

$= 3x^{-1/2} \sin x$

$$u_1' (\cancel{x^{-1/2}}) \left(\cos x + \frac{\sin^2 x}{\cos x} \right) = 3\cancel{x^{-1/2}} \sin x$$

$u_1' (\sin^2 x + \cos^2 x) = 3 \sin x \cos x$

$u_1' = 3 \sin x \cos x$

$u_1 = \frac{3}{2} \sin^2 x + C_1$

$u_2' = \frac{-\sin x}{\cos x} u_1'$

$= -3 \sin^2 x$

$u_2 = -\frac{3x}{2} + \frac{3 \sin x \cos x}{2}$

$$y = u_1 y_1 + u_2 y_2$$
$$= \left(\frac{3}{2}\sin^2 x + c_1\right)\left(x^{-1/2}\sin x\right) + \left(\frac{3}{2}x - \frac{3}{2}\sin x\cos x\right)\left(x^{-1/2}\cos x\right)$$

E.g. 3 Solve $y'' + y = \tan t$

Soln:

$y'' + y = 0$
$r^2 + 1 = 0$
$r^2 = -1$
$r = \pm\sqrt{-1}$
$= \pm i$

$\lambda = 0, \ \mu = 1$

$y_1 = \cos t, \quad y_2 = \sin t$
Let $y = u_1 y_1 + u_2 y_2$
$u_1' y_1 + u_2' y_2 = 0$ (1)
$u_1' y_1' + u_2' y_2' = \tan x$ (2)

From (1) $\quad u_1' = \dfrac{-u_2' y_2}{y_1}$
$= \dfrac{-u_2' \sin t}{\cos t}$

Plug this into (2)

$$\left(\frac{-u_2' \sin t}{\cos t}\right)(-\sin t) + u_2' \cos t = \tan t$$

$u_2' \sin^2 t + u_2' \cos^2 t = \sin t$ Recall: $\tan t = \frac{\sin t}{\cos t}$

$u_2' (\sin^2 t + \cos^2 t) = \sin t$

$u_2' = \sin t$

$u_2 = -\cos t + c_1$

$$u_1' = \frac{-u_2' \sin t}{\cos t}$$

$$= \frac{-\sin^2 t}{\cos t}$$

$$u_1 = \int \frac{-\sin^2 t}{\cos t}\, dt$$

$$= \sin t - \log(1 + \sin t) - \log \cos t + c_2$$

$y = u_1 y_1 + u_2 y_2$

E.g. 4 Solve $x^2 y'' - 3xy' + 4y = x^2 \log x$

Soln:

$x^2 y'' - 3xy' + 4y = 0 \leftarrow$ Euler Eqn

$\alpha = -3, \ \beta = 4$

$r^2 + (\alpha - 1)r + \beta = 0$

$r^2 - 4r + 4 = 0$

$r = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \dfrac{4 \pm 0}{2}$

$= 2$

$y_1 = x^2, \ y_2 = \ln(x) \cdot x^2$

Rewrite the original eqn so that the coefficient of y'' is 1.

$y'' - \dfrac{3y'}{x} + \dfrac{4y}{x^2} = \log x$

$y = u_1 y_1 + u_2 y_2$

$u_1' y_1 + u_2' y_2 = 0 \qquad (1)$

$u_1' y_1' + u_2' y_2' = \log x \qquad (2)$

From (1), $u_1' = \dfrac{-u_2' y_2}{y_1}$

$$= \frac{-u_2' \ln(x) \cdot x^2}{x^2}$$

$$= -u_2' \ln(x)$$

Plug into (2)

$$(-u_2' \ln(x))(2x) + u_2'(x + 2x\ln(x)) = \log x$$

$$u_2'(x + 2x\ln(x) - 2x\ln(x)) = \log x$$

$$u_2'(x) = \log x$$

$$u_2' = \frac{\log x}{x}$$

$$u_2 = \int \frac{\log x}{x}\, dx$$

$$= \frac{\log^2 x}{2} + c_1$$

$$u_1' = -u_2' \ln(x)$$

$$= \frac{-\log^2(x)}{x}$$

$$u_1 = \int \frac{-\log^2 x}{x}\, dx$$

$$= -\frac{\log^3 x}{3} + c_2$$

$$y = u_1 y_1 + u_2 y_2$$

E.g. 5 Solve $y'' - 5y' + 6y = 2e^t$

Soln:

$y'' - 5y' + 6y = 0$

$r^2 - 5r + 6 = 0$

$(r-2)(r-3) = 0$

$r_1 = 2, \ r_2 = 3$

$y_1 = e^{2t}, \quad y_2 = e^{3t}$

Let $y = u_1 y_1 + u_2 y_2$

$u_1' y_1 + u_2' y_2 = 0 \qquad (1)$

$u_1' y_1' + u_2' y_2' = 2e^t \qquad (2)$

From (1), $u_1' = \dfrac{-u_2' y_2}{y_1}$

$$= \frac{-u_2' e^{3t}}{e^{2t}}$$

$$= -u_2' e^t$$

Plug into (2)

$(-u_2' e^t)(2e^{2t}) + u_2'(3e^{3t}) = 2e^t$

$u_2'(3e^{3t} - 2e^{3t}) = 2e^t$

$u_2'(e^{3t}) = 2e^t$

$u_2' = \dfrac{2}{e^{-2t}}$

$$u_2 = \int \frac{2}{e^{-2t}}\, dt$$

$$= \quad - e^{-2t} + c_1$$

$$u_1' = -u_2' e^t$$
$$= \frac{-2}{e^{2t}} e^t$$
$$= \frac{-2}{e^t}$$

$$u_1 = \int \frac{-2}{e^t} \, dt$$
$$= 2e^{-t} + c_2$$

$$y = u_1 y_1 + u_2 y_2$$
$$= (2e^{-t} + c_2)(e^{2t}) + (-e^{-2t} + c_1)(e^{3t})$$